Vorticity and Viscous Dissipation in an Incompressible Flow

Yang-Moon Koh*

(Received April 27, 1993)

The relation between the vorticity and viscous dissipation in an incompressible flow has been investigated. It is shown that the square of relative vorticity with respect to the coordinate system fixed to the container or to the flow at infinity gives the global rate of viscous dissipation, that is, the viscous dissipation is proportional to the volume integral of the square of relative vorticity. Thus, the total rate of viscous dissipation in the flow produced by the motion of a rigid body is proportional to the integral of the square of vorticity taken over the whole space including the region occupied by the volume of the rigid body in which the vorticity is assumed to be the twice of the angular velocity of the body.

Key Words: Vorticity, Viscous Dissipation, Incompressible Flow

1. Introduction

It is one of basic assumptions of fluid mechanics that the viscous stress is independent of the vorticity, but depends only on the rate of deformation tensor. It is also true that the viscous dissipation of mechanical energy, which is the work done in deforming the element of fluid made by the deviatric part of the stress in association with the shearing part of the rate of strain (Batchelor, 1967, p. 153), and the square of vorticity, which is associated with the rotation of the fluid element, are mathematically independent (McCormack and Crane, 1973, §4.2). But the net viscous force on unit volume of an incompressible fluid is proportional to a spatial derivative of the vorticity (Batchelor, 1967, p. 148). Turbulent flow is also due to vorticity intensification through vortex stretching and the dissipation of turbulent kinetic energy is approximately equal to the mean-square vorticity fluctuation (Tennekes and Lumley, 1972, §3.3). In addition, the form of the viscous dissipation function of an incompressible flow,

$$\Phi=\frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right), \quad (1)$$

(here, μ and v_i are the viscosity and i-component of velocity $\boldsymbol{v}$, respectively) is quite similar to that of the square of vorticity,

$$\mu\omega^2=\frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right). \quad (2)$$

All these suggest that there might be some close relations between the vorticity and viscous dissipation.

It seems not only the present author who has thought that the viscous dissipation and vorticity might be related to each other closely. Lamb (1945, p. 581), as an example, has shown that, for an incompressible flow,

$$\int_V \Phi dV=\mu\int_V \omega^2 dV + 2\mu\int_S \left(\boldsymbol{v}\times\boldsymbol{\omega}-\nabla\frac{v^2}{2}\right)\cdot\boldsymbol{n}dS, \quad (3)$$

where the volume integral is taken over the region V occupied by the fluid and the normal $\boldsymbol{n}$ is drawn inwards from the boundary S · If $\boldsymbol{v}=0$ at the boundary, as in the case of a liquid filling a fixed closed vessel, this becomes

$$\int_V \Phi dV=\mu\int_V \omega^2 dV. \quad (4)$$

Thus, we can suppose energy to be dissipated at the rate $\mu\omega^2$ per unit volume (Milne-Thomson, 1968, p. 639). However, generalization of the argument to cases of non-vanishing velocity at the

* Department of Mechanical Engineering, University of Ulsan, Ulsan 680-749, South Korea

boundary seems to have been frustrated by two obvious counterexamples, a rigid-body rotation of a fluid in a vessel and a free-vortex flow around a rotating circular cylinder. Besides, derivation of Eqs. (3) and (4) does not look to be based on physical arguments, but seems purely mathematical.

The purpose of the present study is, thus, to investigate the relation between these two flow properties. We will show that Eq. (4) holds true for any incompressible flow provided that (i) the relative vorticity with respect to the frame of reference fixed to the vessel or to the flow at infinity is used rather than the absolute vorticity and (ii) the region of integration is extended so that it includes the inside of the rigid body where the vorticity is assumed to be equal to the twice of body's angular velocity. Another purpose of the paper is to give the firm physical explanation about why the relation (4) should hold true, that is, why two different physical quantities, ϕ and $\mu\omega^2$, are so closely related. Briefly speaking, the vorticity induces the flow and the viscous dissipation in this induced flow is proportional to the square of vorticity.

2. Changes in the Kinetic Energy of an Incompressible Fluid in Motion

By dot-producting the momentum equation,

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}-\boldsymbol{v}\times\boldsymbol{\omega}\right)=-\nabla\left(p+\frac{\rho v^2}{2}\right)-\mu\nabla\times\boldsymbol{\omega},$$

with veloctiy $\boldsymbol{v}$ and making use of the vector identity,

$$\nabla\cdot(\boldsymbol{v}\times\boldsymbol{\omega})=(\nabla\times\boldsymbol{v})\cdot\boldsymbol{\omega}-\boldsymbol{v}\cdot(\nabla\times\boldsymbol{\omega}),$$

we have the kinetic energy equation,

$$\frac{\partial}{\partial t}\frac{\rho v^2}{2}=-\boldsymbol{v}\cdot\nabla\left(p+\frac{\rho v^2}{2}\right)+\mu\nabla\cdot(\boldsymbol{v}\times\boldsymbol{\omega})-\mu\omega^2,$$

or

$$\frac{d}{dt}\frac{\rho v^2}{2}=-\nabla\cdot(\rho v)+\mu\nabla\cdot(\boldsymbol{v}\times\boldsymbol{\omega})-\mu\omega^2, \quad (5)$$

where p, ρ, and μ represent the pressure, density, and viscosity of the fluid, respectively, and

$$\boldsymbol{\omega}=\nabla\times\boldsymbol{v}.$$

Suppose, now, that a rigid body in an unbounded incompressible viscous fluid which is at rest at infinity is in motion with linear velocity $\boldsymbol{U}(t)$ and angular velocity $\boldsymbol{\Omega}(t)$ (Fig. 1). The rate of change of total kinetic energy of the fluid surrounding the body is then obtained by integrating Eq. (5) over the control volume V between surfaces S_1 and S_2 where the surface S_1 is taken to be coincident with the body surface instantly, and then letting S_2 recede, viz.

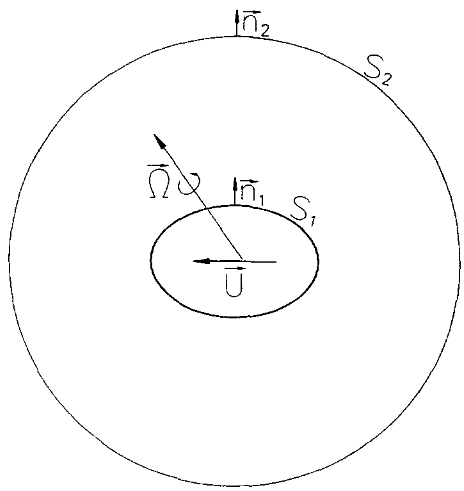

Fig. 1 The control surface used to integrate the kinetic energy equation

$$\frac{d}{dt}\int_V\frac{\rho v^2}{2}dV=\int_{S_1}p\boldsymbol{v}\cdot\boldsymbol{n}_1dS_1-\mu\int_{S_1}(\boldsymbol{v}\times\boldsymbol{\omega})\cdot\boldsymbol{n}_1dS_1-\mu\int_V\omega^2dV. \quad (6)$$

Here, the unit normal $\boldsymbol{n}_1$ is into the control volume (outward from the body).

On S_1, the fluid velocity is equal to that of the body surface, viz.

$$\boldsymbol{v}=\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{r}, \quad (7)$$

and, thus,

$$\int_{S_1}p\boldsymbol{v}\cdot\boldsymbol{n}_1dS_1=\boldsymbol{U}\cdot\int_{S_1}p\boldsymbol{n}_1dS_1+\boldsymbol{\Omega}\cdot\int_{S_1}\boldsymbol{r}\times p\boldsymbol{n}_1dS_1,$$

where r is a position vector to a point on S_1 from the axis of rotation. Also we have

$$\int_{S_1}(\boldsymbol{v}\times\boldsymbol{\omega})\cdot\boldsymbol{n}_1dS_1$$

$$= \boldsymbol{U}\cdot\int_{S_1}(\boldsymbol{\omega}-2\boldsymbol{\Omega})\times\boldsymbol{n}_1 dS_1$$
$$+\boldsymbol{\Omega}\cdot\int_{S_1}\boldsymbol{r}\times\{(\boldsymbol{\omega}-2\boldsymbol{\Omega})\times\boldsymbol{n}_1\}dS_1,$$
$$+2\int_{S_1}(\boldsymbol{v}\times\boldsymbol{\Omega})\cdot\boldsymbol{n}_1 dS_1.$$

But, noting that Eq. (7) also gives velocities of material points of the rigid body and thus applying the divergence theorem to the volume of the body, we have

$$\int_{S_1}(\boldsymbol{v}\times\boldsymbol{\Omega})\cdot\boldsymbol{n}_1 dS_1$$
$$=\int_{V_0}\nabla\cdot(\boldsymbol{v}\times\boldsymbol{\Omega})dV_0=2\Omega^2 V_0,$$

where V_0 is the volume of the rigid body. Thus we have

$$\frac{d}{dt}\int_V\frac{\rho v^2}{2}dV=-(\boldsymbol{U}\cdot\boldsymbol{F}+\boldsymbol{\Omega}\cdot\boldsymbol{T})$$
$$-\mu\Big(\int_V\omega^2 dV+4\Omega^2 V_0\Big), \qquad (8)$$

where

$$\boldsymbol{F}=\boldsymbol{F}_p+\boldsymbol{F}_f,$$
$$\boldsymbol{T}=\boldsymbol{T}_p+\boldsymbol{T}_f,$$
$$\boldsymbol{F}_p=-\int_{S_1}p\boldsymbol{n}_1 dS_1,$$
$$\boldsymbol{F}_f=\mu\int_{S_1}(\boldsymbol{\omega}-2\boldsymbol{\Omega})\times\boldsymbol{n}_1 dS_1$$
$$=\int_{S_1}\tau_s\boldsymbol{t}dS_1,$$
$$\boldsymbol{T}_p=-\int_{S_1}\boldsymbol{r}\times p\boldsymbol{n}_1 dS_1,$$
$$\boldsymbol{T}_f=\mu\int_{S_1}\boldsymbol{r}\times\{(\boldsymbol{\omega}-2\boldsymbol{\Omega})\times\boldsymbol{n}_1\}dS_1$$
$$=\int_{S_1}\boldsymbol{r}\times\tau_s\boldsymbol{t}dS_1,$$

and $\boldsymbol{t}$ is a unit vector tangential to the surface of the body.

As $\boldsymbol{\omega}-2\boldsymbol{\Omega}$ is the vorticity relative to the moving frame of reference fixed to the body, $\mu(\boldsymbol{\omega}-2\boldsymbol{\Omega})\times\boldsymbol{n}=\tau_s\boldsymbol{t}$ is equal to the shear stress at the surface. $\boldsymbol{F}_f$ and $\boldsymbol{T}_f$ are, thus, the resultant force and moment of shear stress acting on the surface of the body, respectively, and $-(\boldsymbol{U}\cdot\boldsymbol{F}+\boldsymbol{\Omega}\cdot\boldsymbol{T})$ is the work done by the body to the surrounding incompressible fulid. Thus, the last term of Eq. (8) should be equal to the dissipation of kinetic energy, viz.

$$\int_V\Phi dV=\mu\Big(\int_V\omega^2 dV+4\Omega^2 V_0\Big). \qquad (9)$$

Eq. (9) can also be got as follows. For an incompressible flow, we have from Eqs. (1) and (2)

$$\Phi=\mu\omega^2+2\mu\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \qquad (10)$$

and also

$$\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}=\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v})$$
$$=\nabla\cdot\Big(-\boldsymbol{v}\times\boldsymbol{\omega}+\nabla\frac{v^2}{2}\Big).$$

Thus, integrating Eq. (10) over the region V, we have Eq. (3) and

$$\int_V\Phi dV=\mu\int_V\omega^2 dV$$
$$-2\mu\int_{S_1}(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{n}_1 dS_1.$$

But (Appendix 1)

$$\int_{S_1}(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{n}_1 dS_1=-2\Omega^2 V_0$$

and Eq. (9) follows immediately. Similarly, for the fluid in a rigid vessel, we have

$$\int_{V_0}\Phi dV=\mu\Big(\int_{V_0}\omega^2 dV-4\Omega^2 V_0\Big), \qquad (11)$$

where V_0 and $\boldsymbol{\Omega}$ are, now, the volume and angular velocity of the vessel, respectively.

In particular, if $\boldsymbol{\Omega}=0$, both Eqs. (9) and (11) are reduced to Eq. (4), that is, the volume integral of the square of vorticity $\mu\int\omega^2 dV$ gets just equal to the total rate of viscous dissipation in the whole flow field. Note also that Eq. (9) is reduced to Eq. (4), if volume integrals are taken over the whole space assuming that the rigid body is replaced by the mass of fluid moving in the same way, viz. assuming that the vorticity is equal to the twice of body's angular velocity in the region occupied by the solid body. Eq. (11) becomes Eq. (4), too, if the relative vorticity $\boldsymbol{\omega}_{rel}=\boldsymbol{\omega}-2\boldsymbol{\Omega}$ with respect to the coordinate system fixed to the vessel is used instead of the absolute vorticity $\boldsymbol{\omega}$. To sum up, the rate of dissipation of kinetic energy of the fluid filling a container which may be infinitely large and have one or several rigid bodies moving in it is given (Eq. (6) holds exactly, when the outer control surface S_2 is the wall of the fixed container and thus $\boldsymbol{v}=0$ there) by

$$\int_V\Phi dV=\mu\int_{V'}\omega^2 dV$$

$$=\mu\left(\int_V\omega^2dV+4\sum\Omega_i^2V_i\right),$$

where V_i is the volume of the i-th body, and V' and V denote, respectively, the inside of the container (the whole region occupied by rigid bodies and fluides) and the region occupied by fluids only. $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}_i$, here, are also the relative vorticity and relative angular velocity with respect to the moving frame of reference fixed to the container.

3. Viscous Dissipation in a Flow Field Induced by Vortices

The previous result that the integral of the square of vorticity is proportional to the rate of viscous dissipation seems, at first sight, to be against the obvious fact that the viscous stress is generated solely by deformation of the fluid and independent of the loacl vorticity. This apparent paradox can be explained by realizing that the vorticity induces the flow and, hence, deformation.

As an example, suppose a Rankine-vortex flow such that

$$v_\theta(r)=\begin{cases}\dfrac{r\omega}{2} & \text{for } r\leq a\\[2mm] \dfrac{a^2\omega}{2r} & \text{for } r>a\end{cases},$$

where v_θ, r, ω and a are values of a circumferential velocity, radial distance, and vorticity and radius of the core of the vortex, respectively. The dissipation function becomes

$$\Phi=\begin{cases}0 & \text{for } r\leq a\\[2mm] \mu\dfrac{a^4\omega^2}{r^4} & \text{for } r>a\end{cases}$$

and integration gives

$$\int\Phi dA=\pi\mu a^2\omega^2=\mu\int\omega^2dA, \tag{12}$$

where integrals are taken over whole plane. We can also show(Appendix 2) that the total dissipation in a combined flow induced by two Rankine vortices of radii a_1, a_2 and vorticities ω_1, ω_2 is equal to $\pi\mu(a_1^2\omega_1^2+a_2^2\omega_2^2)$: the dissipation in the combined flow due to several vortices is simply the sum of dissipations in each flow field due to every single vortex.

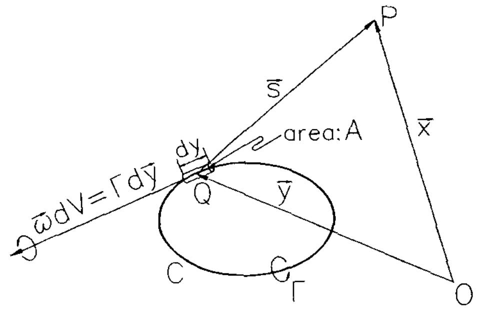

Fig. 2 Calculation of the velocity distribution associated with a ring vortex

Though a bit complicated, it is straightforward to extend the above argument to three-dimensional flows. Suppose a vortex-tube of strength Γ and infinitesimal cross-section A(Fig. 2). It is assumed that the flow is irrotational at infinity and, thus, the tube forms a closed loop C which may pass through the solid body. Let $\boldsymbol{y}$ be a position vector to a point Q of the vortex-tube and $\boldsymbol{\omega}(\boldsymbol{y})$ be the vorticity which is assumed uniform across the cross-section of the vortex-tube. Then, we have

$$\boldsymbol{\omega}(\boldsymbol{y})dV(\boldsymbol{y})=\boldsymbol{\omega}(\boldsymbol{y})Ad\boldsymbol{y}=\Gamma d\boldsymbol{y} \tag{13}$$

and the velocity $\boldsymbol{v}(\boldsymbol{x})$ at a point $P(\boldsymbol{x})$ due to this vortex-tube is (Batchelor, 1967, p. 87)

$$\begin{aligned}\boldsymbol{v}(\boldsymbol{x})&=-\frac{1}{4\pi}\int\frac{\boldsymbol{s}\times\boldsymbol{\omega}(\boldsymbol{y})}{s^3}dV(\boldsymbol{y})\\&=-\frac{\Gamma}{4\pi}\oint_C\frac{\boldsymbol{s}\times d\boldsymbol{y}}{s^3},\end{aligned} \tag{14}$$

where the volume intergral is taken over the vortex-tube and s is the magnitude of the vector $\boldsymbol{s}=\boldsymbol{x}-\boldsymbol{y}$. The strain e_{ij} becomes

$$\begin{aligned}e_{ij}&=-\frac{\Gamma}{8\pi}\oint_C\left(\epsilon_{ilm}\frac{\partial}{\partial x_j}\frac{x_l-y_l}{s^3}+\epsilon_{jlm}\frac{\partial}{\partial x_i}\frac{x_l-y_l}{s^3}\right)dy_m\\&=\frac{\Gamma}{8\pi}\oint_C\left(\epsilon_{ilm}\frac{\partial}{\partial y_j}\frac{x_l-y_l}{s^3}+\epsilon_{jlm}\frac{\partial}{\partial y_i}\frac{x_l-y_l}{s^3}\right)dy_m\end{aligned}$$

and the dissipation function is reduced to

$$\begin{aligned}\Phi=\frac{\mu\Gamma^2}{16\pi^2}\oint_C\oint_C\Big[&\epsilon_{ilm}\epsilon_{ipq}\frac{\partial^2}{\partial y_j\partial y'_j}\frac{(x_l-y_l)(x_p-y'_p)}{s^3s'^3}\\&+\epsilon_{ilm}\epsilon_{jpq}\frac{\partial^2}{\partial y_j\partial y'_i}\frac{(x_l-y_l)(x_p-y'_p)}{s^3s'^3}\Big]dy_mdy'_q,\end{aligned} \tag{15}$$

where $\boldsymbol{y}'$ is another position vector to a point of the vortex-tube and $s'=|\boldsymbol{x}-\boldsymbol{y}'|$.

The total rate of viscous dissipation

$$W=\int \Phi dV(\boldsymbol{x})$$

can now be obtained by integrating Eq. (15) over the whole space. Assuming that the order of integration can be interchanged and using the relation (Appendix 3)

$$\int \frac{(x_i-y_i)(x_j-y_j')}{s^3 s'^3} dV(\boldsymbol{x}) = 2\pi \frac{\partial}{\partial y_i} \frac{y_j-y_j'}{s''}, \tag{16}$$

where $s''=|\boldsymbol{y}-\boldsymbol{y}'|$, we have

$$W=-\frac{\mu\Gamma^2}{8\pi}\oint_C\oint_C\Big[\epsilon_{ilm}\epsilon_{ipq}\frac{\partial^3}{\partial y_j^2\partial y_l}\frac{y_p-y_p'}{s''} + \epsilon_{ilm}\epsilon_{jpq}\frac{\partial^3}{\partial y_j\partial y_i\partial y_l}\frac{y_p-y_p'}{s''}\Big]dy_m dy_q', \tag{17}$$

The last integral of Eq. (17) is zero and we have

$$W=-\frac{\mu\Gamma^2}{8\pi}\Big[\oint_C\oint_C\frac{\partial^3}{\partial y_j^2\partial y_l}\frac{y_l-y_l'}{s''}dy_m dy_m' - \oint_C\oint_C\frac{\partial^3}{\partial y_j^2\partial y_l}\frac{y_m-y_m'}{s''}dy_m dy_l'\Big]. \tag{18}$$

The last integral of Eq. (18) is zero again and

$$\frac{\partial^3}{\partial y_j^2\partial y_l}\frac{y_l-y_l'}{s''}=\frac{\partial^2}{\partial y_j^2}\frac{2}{s''}=-8\pi\delta(\boldsymbol{y}-\boldsymbol{y}'),$$

where $\delta(\boldsymbol{y}-\boldsymbol{y}')$ is a Dirac δ-function. Thus, using Eq. (13), we have

$$\begin{aligned} W&=\mu\Gamma^2\oint_C\oint_C\delta(\boldsymbol{y}-\boldsymbol{y}')dy_m dy_m' \\ &=\mu\iint\omega_m(\boldsymbol{y})\omega_m(\boldsymbol{y}')\delta(\boldsymbol{y}-\boldsymbol{y}')dV(\boldsymbol{y})dV(\boldsymbol{y}') \\ &=\mu\int\boldsymbol{\omega}(\boldsymbol{y})\cdot\boldsymbol{\omega}(\boldsymbol{y})dV(\boldsymbol{y}), \end{aligned} \tag{19}$$

where volume integrals are taken over the vortex-tube. In other words, the viscous dissipation in the flow induced by an infnitesimal vortex is proportional to the volume integral of the square of vorticity taken over the vortex.

The combined velocity due to two vortices C^I and C^{II} is the sum of velocities induced by each vortex and so is the strain ϵ_{ij} of the combined flow, that is,

$$\epsilon_{ij}=\epsilon_{ij}^I\epsilon_{ij}^{II},$$

where ϵ^I and ϵ^{II} are strains due to C^I and C^{II}, respectively. Thus we have

$$\epsilon_{ij}\epsilon_{ij}=\epsilon_{ij}^I\epsilon_{ij}^I+2\epsilon_{ij}^I\epsilon_{ij}^{II}+\epsilon_{ij}^{II}\epsilon_{ij}^{II}.$$

But the same reasoning to get Eq. (19) leads to that the integral of the cross-product term is zero, i.e.,

$$\int\epsilon_{ij}^I\epsilon_{ij}^{II}dV(\boldsymbol{x})=0,$$

and we have the same relation (19) for the combined flow, too, with the volume integral taken over all the vortices. But it is irrotational outside the vortices and we have Eq. (4) again, viz.

$$W=\int\Phi dV(\boldsymbol{x})=\mu\int\omega^2 dV(\boldsymbol{x}),$$

where volume integrals are now taken over the whole space.

4. Discussions and Concluding Remarks

It seems not necessary to discuss viscous dissipations in a rigid-body rotation of the fluid in a vessel and a free-vortex flow around a rotating circular cylinder any more. But one may raise a question about viscous dissipations in potential flows. Clearly, irrotationality does not imply the absence of viscous stresses (Kundu, 1990, p. 124) nor the absence of viscous dissipations. But, the only potential flow known to the present author that satisfies the no-slip condition at the solid wall is the free-vortex flow around a rotating circular cylinder which has already been discussed in depth. To satisfy the no-slip condition, most potential flows are accompanied by shear flows with non-vanishing vorticity such as in boundary layers and wakes. The velocity $\boldsymbol{v}$ of an incompressible fluid which is consistent with specified values of vorticity $\boldsymbol{\omega}$ at each point of the fluid can be written (Batchelor, 1967, Sec. 2.4) as

$$\boldsymbol{v}=\boldsymbol{v}_v+\nabla\phi, \tag{20}$$

where $\boldsymbol{v}_v$ is given by Eq. (14) with the volume integral taken over the region occupied by the fluid. The velocity potential ϕ is to satisfy the boundary condition. It should also be assumed that the fluid, the vorticity distribution and, hence, the region of integration extend beyond the actual boundary to the inside of the body so that all the vortex lines form closed loops in the extended region, when the specified vorticity has non-zero normal component at some points of the actual boundary of the fluid. But, imagining that the body is replaced by the mass of fluid moving like the rigid body(the no-slip condition assures that the velocity varies continuosly across the actual boundary of the fluid), we can see that Eq. (14), the volume integral being taken over the extended region which includes the volume of the body, gives the velocity $\boldsymbol{v}$ and, hence, $\nabla\phi$ in Eq. (20) can be taken to be zero. In other words, the vorticity distribution in the shear layer and inside of the volume of the body determines the whole velocity field including that of the external potential flow, and the integral of the square of vorticity over the extended region gives the total rate of viscous dissipation in the whole flow field including that in the region of irrotational flow.

Neither the vorticity at a point nor the square of it is associated with the local rate of viscous dissipation at that point. As claimed frequently, the viscous dissipation, which is associated with the deformation of the element of fluid, and the square of vorticty, which is associated with the rigid-body rotation of the fluid element, are not only two different physical quantities but also are mathematically independent. But the relative vorticity, i.e., the relative rotation of an incompressible fluid with respect to the coordinate system fixed to the container or to the flow at infinity always induces deformation. In other words, if there is a relatvie motion of an incompressible fluid with respect to the container or to the flow at infinity, then there always exists the relative vorticity, $\boldsymbol{\omega}_{rel}$ and the integral $\mu\int\omega_{rel}^2 dV$ taken over the extended region including the volume of the rigid body is equal to the total viscous dissipation. Briefly speaking, $\mu\omega_{rel}^2$ gives the global rate of viscous dissipation, while the dissipation funtion Φ gives the local rate, and their volume integrals over the extended region should be the same.

Acknowledgement

The author is grateful to Prof. P. Bradshaw of Standford University for reading the manuscript and giving valuable comments.

References

Batchelor, G. K., 1967, *An Introduction to Fluid Dynamics,* Cambridge University Press.

Dwight, H. B., 1961, *Tables of Integrals and Other Mathematical Data,* 4th ed., The Macmillan Company.

Kundu, P. K., 1990, *Fluid Mechanics,* Academic Press.

Lamb, H., 1945, *Hydrodynamics,* 6th ed., Dover Publications.

McCormack, P. D. and Crane, L., 1973, *Physical Fluid Dynamics,* Academic Press.

Milne-Thomson, L. M., 1968, *Theoretical Hydrodynamics,* 5th ed., The MacMillan Company.

Tennekes, H. and Lumley, J. L., 1972, *A First Course in Turbulence,* The MIT Press.

Appendix A
Evaluation of the Integral

$$\int_{S_1}(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{n}_1 dS_1$$

Let

$$\boldsymbol{v}=\boldsymbol{v}_{rel}+\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{r}$$

where $\boldsymbol{r}$ is a position vector from the origin of the moving coordinate system fixed to the body and $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are the linear and angular velocity of the moving frame of reference, respectively. Then, employing the no-slip condition

$$\boldsymbol{v}_{rel}=0 \quad \text{on } S_1$$

and the vector identity

$$\boldsymbol{a}\cdot\nabla(\boldsymbol{\Omega}\times\boldsymbol{r})=\boldsymbol{\Omega}\times\boldsymbol{a},$$

where $\boldsymbol{a}$ is an arbitrary vector, we have

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v}=(\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\nabla\boldsymbol{v}_{rel} +\boldsymbol{\Omega}\times(\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{r})$$

on the surface S_1.

But $\boldsymbol{v}_{rel}\cdot\boldsymbol{n}_1$, the normal component of relative velocity with respect to the surface S_1, is a quadratic function of the distance from S_1 and, thus,

$$[(\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\nabla\boldsymbol{v}_{rel}]\cdot\boldsymbol{n}_1=0$$

on S_1. Also we have

$$\int_{S_1}(\boldsymbol{\Omega}\times \boldsymbol{U})\cdot \boldsymbol{n}_1 dS_1 = (\boldsymbol{\Omega}\times \boldsymbol{U})\cdot \int_{S_1}\boldsymbol{n}_1 dS_1 = 0$$

and

$$\int_{S_1}[\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times \boldsymbol{r})]\cdot \boldsymbol{n}_1 dS_1 = \int_{V_0}\nabla\cdot[\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times \boldsymbol{r})]dV_0 = -2\Omega^2 V_0,$$

where V_0 is the volume of the body. Thus, we have

$$\int_{S_1}(\boldsymbol{v}\cdot\nabla \boldsymbol{v})\cdot \boldsymbol{n}_1 dS_1 = -2\Omega^2 V_0.$$

Appendix B
Viscous Dissipation in a Flow Field Induced by Two Rankine Vortices

The velocity $\boldsymbol{v}$ at a point $P(x, y)$ (Fig. 3) induced by two line vortices of strengths Γ_1, Γ_2 at points $(\pm l, 0)$ is

$$\boldsymbol{v} = \frac{\Gamma_1}{2\pi}\frac{-y\boldsymbol{i}+(x-l)\boldsymbol{j}}{(x-l)^2+y^2} + \frac{\Gamma_2}{2\pi}\frac{-y\boldsymbol{i}+(x+l)\boldsymbol{j}}{(x+l)^2+y^2}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the x- and y-direction, respectively. The dissipation function becomes

$$\frac{\Phi}{\mu} = \frac{\Gamma_1^2}{\pi^2}\frac{1}{[(x-l)^2+y^2]^2} + \frac{\Gamma_2^2}{\pi^2}\frac{1}{[(x+l)^2+y^2]^2} + \frac{2\Gamma_1\Gamma_2}{\pi^2}\frac{4(x-l)(x+l)y^2+[(x-l)^2-y^2][(x+l)^2-y^2]}{[(x-l)^2+y^2]^2[(x+l)^2+y^2]^2}.$$

The first two terms on the right-hand side are dissipation functions for each line-vortex flow and the last one, denoted by Φ_{III}, is due to the interaction of two vortex-flows.

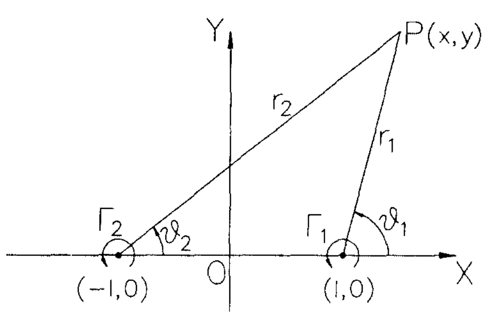

Fig. 3 Calculation of the velocity field induced by two line vortices

To integrate Φ_{III}, we can use the bipolar coordinate system(Fig. 3):

$$\xi = \theta_1 - \theta_2, \qquad \eta = \ln\frac{r_1}{r_2},$$

$$x = \frac{l\sinh\eta}{\cosh\eta - \cos\xi}, \quad y = \frac{l\sin\zeta}{\cosh\eta - \cos\xi}.$$

The scale factors are

$$h_\xi = h_\eta = \frac{l}{\cosh\eta - \cos\xi}$$

and Φ_{III} becomes

$$\Phi_{III} = \frac{\Gamma_1\Gamma_2}{2\pi^2 l^4}(\cosh\eta - \cos\xi)^2\cos 2\xi.$$

Thus, we have

$$\iint \Phi_{III}dxdy = \frac{\Gamma_1\Gamma_2}{2\pi^2 l^2}\iint \cos 2\xi d\xi d\eta = 0,$$

where the integrals are taken over the whole plane excluding the cores of vortices, that is, the region outside of two circles of constant r_1/r_2 centred at $(\pm l, 0)$. Thus, the total dissipation in a combined flow due to two Rankine vortices is simply the sum of dissipations in each Rankine-vortex flow.

Appendix C
Derivation of Eq. (16)

By putting

$$\boldsymbol{a} = \frac{\boldsymbol{y}-\boldsymbol{y}'}{2}, \quad \boldsymbol{t} = \boldsymbol{x} - \frac{\boldsymbol{y}+\boldsymbol{y}'}{2},$$

we have

$$\boldsymbol{s} = \boldsymbol{x}-\boldsymbol{y} = \boldsymbol{t}-\boldsymbol{a}, \quad \boldsymbol{s}' = \boldsymbol{x}-\boldsymbol{y}' = \boldsymbol{t}+\boldsymbol{a},$$

and the left-hand side of Eq. (16), denoted by I_{ij}, becomes

$$I_{ij} = \int\frac{(t_i-a_i)(t_j+a_j)}{s^3 s'^3}dV(\boldsymbol{t}).$$

Let $\boldsymbol{e}_i$ be a unit vector in the i-direction of the coordinate system and $\boldsymbol{e}_i'$s be another set of base vectors such that $\boldsymbol{e}_3'$ is in the direction of $\boldsymbol{a}$. Then, putting

$$\boldsymbol{t} = r(\cos\theta \boldsymbol{e}_1' + \sin\theta \boldsymbol{e}_2') + z\boldsymbol{e}_3',$$

where θ is the azimuthal angle about the $\boldsymbol{e}_3'$-axis measured from the $\boldsymbol{e}_1'$-axis and r is the distance from the $\boldsymbol{e}_3'$-axis, and integrating with respect to θ first, we have

$$I_{ij}=\pi\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{2(z^2-a^2)l_{3i}l_{3j}+r^2(\delta_{ij}-l_{3i}l_{3j})}{[r^2+(z-a)^2]^{3/2}[r^2+(z+a)^2]^{3/2}}rdrdz$$

where $l_{ij}=\boldsymbol{e}_i'\cdot\boldsymbol{e}_j$. By substituting the variable of integration r by r^2 and using the tabulated formulae(Dwight, 1961, p. 214)

$$\int_0^{\infty}\frac{dx}{(ax^2+2bx+c)^{3/2}}=\frac{1}{b\sqrt{c}+c\sqrt{a}}$$

and

$$\int_0^{\infty}\frac{xdx}{(ax^2+2bx+c)^{3/2}}=\frac{1}{a\sqrt{c}+b\sqrt{a}},$$

we have

$$I_{ij}=\pi\frac{\delta_{ij}-l_{3i}l_{3j}}{a}=2\pi\frac{\delta_{ij}-l_{3i}l_{3j}}{|\boldsymbol{y}-\boldsymbol{y}'|}.$$

But

$$l_{3i}=\boldsymbol{e}_3'\cdot\boldsymbol{e}_i=\frac{(\boldsymbol{y}-\boldsymbol{y}')}{|\boldsymbol{y}-\boldsymbol{y}'|}\cdot\boldsymbol{e}_i=\frac{y_i-y_i'}{|\boldsymbol{y}-\boldsymbol{y}'|}$$

and the integral becomes

$$I_{ij}=2\pi\left[\frac{\delta_{ij}}{|\boldsymbol{y}-\boldsymbol{y}'|}-\frac{(y_i-y_i')(y_j-y_j')}{|\boldsymbol{y}-\boldsymbol{y}'|^3}\right]$$
$$=2\pi\frac{\partial}{\partial y_i}\frac{(y_j-y_j')}{|\boldsymbol{y}-\boldsymbol{y}'|}.$$